

The Betti poset in monomial resolutions

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Abstract

Let P be a finite partially ordered set with unique minimal element $\hat{0}$. We study the Betti poset of P , created by deleting elements $q \in P$ for which the open interval $(\hat{0}, q)$ is acyclic. Using basic simplicial topology, we demonstrate an isomorphism in homology between open intervals of the form $(\hat{0}, p) \subset P$ and corresponding open intervals in the Betti poset. Our motivating application is that the Betti poset of a monomial ideal's lcm-lattice encodes both its \mathbb{Z}^d -graded Betti numbers and the structure of its minimal free resolution. In the case of rigid monomial ideals, we use the data of the Betti poset to explicitly construct the minimal free resolution. Subsequently, we introduce the notion of rigid deformation, a generalization of Bayer, Peeva, and Sturmfels' generic deformation.

Introduction

There has been a great deal of work on the problem, posed by Kaplansky in the early 1960s, of constructing a minimal free resolution of a monomial ideal M in a polynomial ring R . While there are several methods for computing the Betti numbers, a method for constructing the maps in a minimal resolution remains elusive except for specific classes of ideals.

Our contributions to this problem are as follows. In Theorem 2.6, we give a construction for the minimal free resolution of a rigid monomial ideal. We follow this by defining *rigid deformations*, which leverage this construction to give minimal free resolutions of non-rigid monomial ideals. This program is easily shown successful in the case of simplicial resolutions, which we verify in Theorem 2.12. Combined with knowledge of join-preserving maps between finite atomic lattices, this gives a structural understanding of the pathways and obstructions to minimally resolving monomial ideals by discerning the resolution structure of combinatorially similar ideals.

Our overall program mirrors that of generic monomial ideals [1, 11], which are known to have their minimal resolutions supported on the Scarf simplicial complex Δ_M . In this case, the maps in the minimal resolution are formed by \mathbb{Z}^d -grading the maps in the algebraic chain complex of Δ_M . Bayer, Peeva, and Sturmfels [1] also introduce the notion of generic deformation of exponents,

which gives a (generally non-minimal) simplicial resolution for every ideal, supported on the Scarf complex of the target generic ideal. The benefit of our approach is an ability to describe (and in turn deform to) resolutions whose structure is more complicated than the simplicial resolutions of generic ideals.

The second author establishes the motivation for our approach by showing in [9] that generic deformation has blind spots. First, a poor choice of deformed exponents can result in a generic ideal whose minimal resolution is as large as possible – the Taylor complex [15]. Moreover, there exist finite atomic lattices whose ideals have simplicial resolutions, but are not the result of any generic deformation. In particular, there exist even monomial ideals of projective dimension two whose minimal resolutions have a very basic simplicial structure, but which cannot be attained by the process of generic deformation. Our approach overcomes these deficiencies by taking advantage of the lattice structure of the set of finite atomic lattices on a fixed number of atoms due to Phan[14]. In particular, we investigate the role that rigid ideals take in this lattice.

The main tool in our description of the minimal resolution of rigid ideals is the order-theoretic perspective taken by the first author in [4]. The technique presented therein creates a complex of vector spaces from the homology of intervals in a poset and yields minimal resolutions in some cases. It recovers the resolutions given by several known construction methods as being supported on a relevant poset, including the Scarf resolution for generic ideals.

To describe the resolution structure of a rigid ideal, we identify the combinatorial object which encodes the unique \mathbb{Z}^d -graded bases in a rigid ideal's minimal resolution, extending our initial results [5] on rigid ideals. Where the Scarf complex is the unique object supporting the minimal resolution of a generic ideal, the *Betti poset* plays the same role for a rigid ideal. The Betti poset is the subposet of an ideal's lcm-lattice which does not contain the homologically irrelevant data present in the lcm-lattice. We provided initial examples of the utility of the Betti poset in [5], and this object has been subsequently studied in [16] using techniques of category theory.

Fundamental progress comes in Section 1 by proving general statements about the Betti poset of a poset P . Specifically, Theorem 1.4 uses classical techniques from simplicial homology to establish an isomorphism between the homology of certain open intervals of a poset P and those analogous intervals appearing in its Betti poset.

In Section 2, we turn our attention to the lcm-lattice of a monomial ideal M and its Betti poset B_M . First, we establish Theorem 2.1 by using basic relabeling techniques to show that the isomorphism class of B_M completely determines its minimal free resolution. Our proof avoids the functorial techniques of [16].

Our main result is Theorem 2.6, which states that when an ideal is rigid, the Betti poset of its lcm-lattice supports the minimal free resolution. Section 3 contains the technical proof of Theorem 2.6.

With results on rigid ideals in hand, we introduce the notion of rigid deformation by appealing to the structure of $\mathcal{L}(n)$, the lattice of finite atomic lattices introduced by Phan [14]. Indeed, as a corollary to Theorem 2.6, we see that for every monomial ideal M whose lcm-lattice is comparable in $\mathcal{L}(n)$ to that of a

rigid ideal, the Betti poset of the rigid ideal supports the minimal resolution of any monomial ideal whose Betti poset is isomorphic to B_M . This generalizes the equivalent Theorem 2.1 and the results of [16]. For a non-rigid ideal, there may be several rigid ideals whose Betti poset supports a minimal resolution. Conversely, we present an example of a non-rigid ideal which admits no rigid deformation.

Throughout the paper, all posets are finite with a unique minimal element $\hat{0}$. In order to focus attention on the homological properties of their order complexes, when considering the topological and homological properties of the order complex $\Delta(P)$ we write P in its place when the context is clear.

1 Homological properties of the Betti poset

In the category of simplicial complexes, the notion of vertex deletion is well-studied. We review the analogous concept for deleting elements from a poset P . Using this process, we show that under certain assumptions, deleting particular elements from a fixed open interval in a poset induces an isomorphism in homology.

Let p be an element of the poset P . We refer to the poset $P \setminus \{p\}$ as the *deletion of p from P* since in the order complex of (intervals of) P , this operation corresponds to the topological notion of vertex deletion. The idea of this procedure is related to the one described by Flöystad [7].

In what follows, we examine the reduced homology of order complexes of posets. For a poset P , we write $h_i = h_i(P) = \dim_{\mathbb{k}} \tilde{H}_i(\Delta(P), \mathbb{k})$ for the \mathbb{k} -vector space dimension of the homology of the order complex of the poset $P \setminus \{\hat{0}\}$. When $p \in P$, for notational simplicity we write $\Delta_p = \Delta_p^P = \Delta(\hat{0}, p)$ for the order complex of the associated open interval in P . As is standard, we say that an element $x \in P$ is *covered* by y (which we write as $x \lessdot y$) when $x < y$ and there exists no $z \in P$ such that $x < z < y$. For those $p \in P$ with the property that $h_i(\Delta_p) = 0$ for every i , we say that p is a *non-contributor* to the homological data of P .

We now apply the notions of simplicial deletion, link, and star to the order complex of a poset. Focusing on the order complex Δ_q and a poset element $p \in (\hat{0}, q)$ recall

$$\begin{aligned} \text{del}_{\Delta_q}(p) &= \{\sigma \in \Delta_q : p \notin \sigma\}, \\ \text{link}_{\Delta_q}(p) &= \{\sigma \in \Delta_q : \sigma \cup \{p\} \in \Delta_q\}, \text{ and} \\ \text{star}_{\Delta_q}(p) &= \{\sigma \in \Delta_q : p \in \sigma\}. \end{aligned}$$

Our interest in $\text{link}_{\Delta_q}(p)$ comes from its relationship to $\text{del}_{\Delta_q}(p)$ and $\text{star}_{\Delta_q}(p)$, which will be of use in analyzing the homological data in P .

When restricting to the subposet $(\hat{0}, q) \subset P$, the complex $\text{link}_{\Delta_q}(p)$ is the simplicial join $\Delta(0, p) * \Delta(p, q)$. Using this notion, we prove the following.

Lemma 1.1. *Let P be a poset and fix $q \in P$. If $p < q \in P$ has the property that $h_j(\Delta_p) = 0$ for every j , then there is an isomorphism in homology $\tilde{H}_i(\Delta_q, \mathbb{k}) \cong \tilde{H}_i(\text{del}_{\Delta_q}(p), \mathbb{k})$.*

Proof. Take $q \in P$ and recall that for any element $p < q$ in $(\hat{0}, q)$, we have $\text{link}_{\Delta_q}(p) = \Delta_p * \Delta(p, q)$. Using the Kunneth formula, the reduced homology of $\text{link}_{\Delta_q}(p)$ is

$$\tilde{H}_k(\text{link}_{\Delta_q}(p)) = \bigoplus_{i+j=k-1} \tilde{H}_i(\Delta_p) \otimes \tilde{H}_j(\Delta(p, q)).$$

Since p is assumed to be a non-contributor, $\tilde{H}_i(\Delta_p) = 0$ for every i and therefore $\tilde{H}_k(\text{link}_{\Delta_q}(p)) = 0$ for every k .

Next, consider the Mayer-Vietoris sequence in reduced homology for the triple

$$(\text{del}_{\Delta_q}(p), \text{star}_{\Delta_q}(p), \Delta_q).$$

Since $\text{link}_{\Delta_q}(p) = \text{del}_{\Delta_q}(p) \cap \text{star}_{\Delta_q}(p)$ has homology which vanishes in every dimension, the Mayer-Vietoris sequence

$$\begin{aligned} \cdots \rightarrow \tilde{H}_i(\text{link}_{\Delta_q}(p)) \rightarrow \tilde{H}_i(\text{del}_{\Delta_q}(p)) \oplus \tilde{H}_i(\text{star}_{\Delta_q}(p)) \\ \rightarrow \tilde{H}_i(\Delta_q) \rightarrow \tilde{H}_{i-1}(\text{link}_{\Delta_q}(p)) \rightarrow \cdots \end{aligned}$$

reduces for each ℓ to

$$0 \rightarrow \tilde{H}_\ell(\text{del}_{\Delta_q}(p)) \oplus \tilde{H}_\ell(\text{star}_{\Delta_q}(p)) \xrightarrow{\psi_\ell} \tilde{H}_\ell(\Delta_q) \rightarrow 0$$

Hence, the map ψ_ℓ is an isomorphism. Furthermore, the simplicial complex $\text{star}_{\Delta_q}(p)$ is a cone with apex p and therefore $\tilde{H}_\ell(\text{star}_{\Delta_q}(p)) = 0$ for all ℓ . Thus, $\tilde{H}_\ell(\text{del}_{\Delta_q}(p), \mathbb{k}) \cong \tilde{H}_\ell(\Delta_q, \mathbb{k})$ for all ℓ . \square

When Δ_p has trivial homology, the isomorphism between the homology of the order complex of the original open interval $(\hat{0}, q)$ and the homology of the order complex of the poset $(\hat{0}, q) \setminus \{p\}$ suggests the following definition

Definition 1.2. The *Betti poset* of a poset P is the subposet consisting of all homologically contributing elements,

$$B(P) = \{q \in P \mid \tilde{H}_i(\Delta_q) \neq 0 \text{ for at least one } i\}.$$

The name *Betti poset* is motivated by the study of minimal resolutions of monomial ideals. This object was introduced in [5] and studied in [16]. We specialize to monomial ideal setting in Section 2. In particular, if P is a finite atomic lattice, then it is poset-isomorphic to the lcm-lattice of a monomial ideal. In this context, the elements of the Betti poset of P consist of those multidegrees of a monomial ideal which contribute Betti numbers in the minimal free resolution. Before turning to this application, we prove several general facts about Betti posets.

Corollary 1.3. Suppose $p \in P$ is a noncontributor, i.e. that $h_i(\Delta_p) = 0$ for all i . For $P' = P \setminus \{p\}$ with the induced partial ordering, $B(P) = B(P')$.

Proof. Suppose $p \in P$ has the property that $h_i(\Delta_p^P) = 0$ for every i . For those $q \in P$ such that $q < p$, or when q is not comparable to p , the interval $(\hat{0}, q)$ is the same in both P and P' . As such, the isomorphism on order complexes induces an isomorphism of homology. Thus, if $q < p$ then $q \in B(P)$ if and only if $q \in B(P')$.

For $q > p$, then according to Lemma 1.1, $\tilde{H}_i(\Delta_q^P) \cong \tilde{H}_i(\text{del}_{\Delta_q^P}(p))$ for every i . However, $\text{del}_{\Delta_q^P}(p) = \Delta_q^{P'}$ and therefore $\tilde{H}_i(\Delta_q^P) \cong \tilde{H}_i(\Delta_q^{P'})$. Hence, if $q > p$ then $q \in B(P)$ if and only if $q \in B(P')$. \square

Theorem 1.4. *Let P be a poset and $B(P)$ its Betti poset. For each $q \in B(P)$ we have an isomorphism of \mathbb{k} -vector spaces $\tilde{H}_i(\Delta_q^P) \cong \tilde{H}_i(\Delta_q^{B(P)})$.*

Proof. By induction on the number of non-contributing elements in P .

The base case where P and $B(P)$ differ by only one non-contributing element p , is a special case of Corollary 1.3. Thus, for every $q \in B(P)$, we have $\tilde{H}_i(\Delta_q^P) \cong \tilde{H}_i(\Delta_q^{B(P)})$.

Let $k > 1$ and suppose that for any poset P' which has fewer than k non-contributing elements, $\tilde{H}_i(\Delta_q^{P'}) \cong \tilde{H}_i(\Delta_q^{B(P')})$ for every $q \in B(P')$. Let P be a poset which has k non-contributing elements. Write p for a non-contributor of P , so that we have $P = P' \cup \{p\}$ for some p . Taking $q \in B(P)$ we have two possibilities, similar to the proof of Corollary 1.3.

First, if $q < p$ or if q is not comparable to p , then $(\hat{0}, q)$ is the same in both P' and P , inducing an isomorphism in homology $\tilde{H}_i(\Delta_q^P) \cong \tilde{H}_i(\Delta_q^{P'})$. Using the induction hypothesis in concert with the fact that Corollary 1.3 guarantees $B(P) = B(P')$, we have

$$\tilde{H}_i(\Delta_q^P) \cong \tilde{H}_i(\Delta_q^{P'}) \cong \tilde{H}_i(\Delta_q^{B(P')}) \cong \tilde{H}_i(\Delta_q^{B(P)}).$$

Alternately, if $q > p$ then Lemma 1.1 applies, and viewing each element in the larger poset P , we have $\tilde{H}_i(\Delta_q^P) \cong \tilde{H}_i(\text{del}_{\Delta_q^P}(p))$. Since $\text{del}_{\Delta_q^P}(p) = \Delta_q^{P'}$, then we obviously have $\tilde{H}_i(\text{del}_{\Delta_q^P}(p)) \cong \tilde{H}_i(\Delta_q^{P'})$. The induction hypothesis now applies to P' , a poset with fewer than k non-contributors. Together with the equality of posets $B(P) = B(P')$ guaranteed by Corollary 1.3, we have

$$\tilde{H}_i(\Delta_q^P) \cong \tilde{H}_i(\text{del}_{\Delta_q^P}(p)) \cong \tilde{H}_i(\Delta_q^{P'}) \cong \tilde{H}_i(\Delta_q^{B(P')}) \cong \tilde{H}_i(\Delta_q^{B(P)}),$$

which completes the proof. \square

2 Resolutions of monomial ideals

We now use the results of Section 1 to study the minimal free resolution of a monomial ideal M in a polynomial ring R . Recall that the lcm-lattice of M is the set L_M of least common multiples of the n minimal generators of M , with

minimal element $1 \in R$ and ordering given by divisibility. For the remainder of the paper, we consider the Betti poset of an lcm-lattice, which we denote $B_M = B(L_M)$.

With the appropriate notions established, we state the first commutative algebra result of this paper.

Theorem 2.1. *Suppose $M \subset R = \mathbb{k}[x_1, \dots, x_d]$ and $N \subset S = \mathbb{k}[y_1, \dots, y_t]$ are monomial ideals such that $B_M \cong B_N$. A minimal resolution of M can be relabeled to give the minimal resolution of N .*

Proof. Suppose $(\mathcal{F}_M, \partial^M)$ is a minimal free resolution of M . We exploit the hypothesized isomorphism on Betti posets $g : B_M \rightarrow B_N$ to write a minimal resolution for N .

Since $B_M \cong B_N$, then for every $\mathbf{x}^{\mathbf{a}} \in B_M$, there exists a corresponding monomial $g(\mathbf{x}^{\mathbf{a}}) \in B_N$ of multidegree \mathbf{b} . For every such $g(\mathbf{x}^{\mathbf{a}}) \in B_N$, write $V_{\mathbf{b}}^i$ as a rank i vector space over \mathbb{k} with basis $\{e_{\mathbf{b}}^j\}_{j=1}^i$. The vector spaces $V_{\mathbf{b}}^i$ are in one-to-one correspondence with the free modules of \mathcal{F}_M , with $V_{\mathbf{b}}^i$ appearing in homological degree r since $\tilde{H}_{r-2}(\Delta(1, \mathbf{x}^{\mathbf{a}}))$ is nonzero.

To define the differential of \mathcal{F}_N , suppose the differential of \mathcal{F}_M takes the form

$$\partial^M(e_{\mathbf{a}}) = \sum c_{\mathbf{a}, \mathbf{a}'} \cdot \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{x}^{\mathbf{a}'}} \cdot e_{\mathbf{a}'}$$

where $c_{\mathbf{a}, \mathbf{a}'} \in \mathbb{k}$, the bases $e_{\mathbf{a}}$ and $e_{\mathbf{a}'}$ are respective generators for free modules in homological degree r and $r-1$, and $\mathbf{x}^{\mathbf{a}'} < \mathbf{x}^{\mathbf{a}} \in B_M$. The differential of the complex $\mathcal{F}_M \otimes R/(x_1 - 1, \dots, x_d - 1)$ is

$$\delta^M(e_{\mathbf{a}}) = \sum c_{\mathbf{a}, \mathbf{a}'} \cdot e_{\mathbf{a}'}.$$

We now relabel this complex and its differential using the data of the Betti poset B_N . For every $g(\mathbf{x}^{\mathbf{a}}) \in B_N$ of multidegree \mathbf{b} , define a free module with appropriate shift in multidegree, $S(-\mathbf{b})$. Certainly, each summand of the vector space $V_{\mathbf{b}}^i$ appearing in our complex corresponds to exactly one of the free modules whose shift is $-\mathbf{b}$. Next, define the action of the differential of \mathcal{F}_N as

$$\partial^N(e_{\mathbf{b}}) = \sum c_{\mathbf{a}, \mathbf{a}'} \cdot \frac{g(\mathbf{x}^{\mathbf{a}})}{g(\mathbf{x}^{\mathbf{a}'})} \cdot e_{\mathbf{b}'}$$

where \mathbf{b}', \mathbf{b} are the multidegrees of the monomials corresponding to the comparison $g(\mathbf{x}^{\mathbf{a}'}) < g(\mathbf{x}^{\mathbf{a}}) \in B_N$.

The sequence \mathcal{F}_N certainly has the minimum number of free modules, since they were created using the data of the Betti poset B_N . Furthermore, \mathcal{F}_N is a resolution of S/N if and only if the subcomplex of $\mathcal{F}_N(\leq m)$ is exact for every $m \in L_N$. The exactness of \mathbb{Z}^t -graded strands is certainly satisfied, since the \mathbb{Z}^t -degrees of \mathcal{F}_N are in one-to-one correspondence with those of \mathcal{F}_M through the isomorphism $B_M \cong B_N$. Hence, \mathcal{F}_N is a minimal free resolution of S/N . \square

Remark 2.2. The conclusion of Theorem 2.1 is different than part (3) of Theorem 4.3 in [13], which also addresses minimal resolutions. Their result assumes both an equality of total Betti numbers and the existence of a map between lcm-lattices which is a least common multiple preserving bijection on atoms. Under their assumption, one relabels a resolution of the source lattice's monomial ideal to create a resolution of the target lattice's monomial ideal.

Our assumption of an isomorphism on Betti posets is more restrictive than one assuming equality of total Betti numbers. Indeed, ideals with the same total Betti numbers can have distinct Betti posets. However, our stronger assumption removes the need for a reduction map between lcm-lattices, while maintaining the ability to relabel minimal resolutions. In particular, monomial ideals which share the same Betti poset need not have lcm-lattices which are related by a reduction map.

Example 2.3. To illustrate the previous remark consider the following example. Let

$$M = (b^2ce^2f^2, cde^2f^2, ade^2f^2, abef, ab^2cdf, ab^2cde)$$

and let

$$N = (bce^2f^2, cde^2f^2, are^2f^2, a^2be^2f^2, a^2bcdf, a^2bcde)$$

be two ideals in $k[a, b, c, d, e, f]$. Their lcm-lattices (with the Betti poset elements marked as bold dots) are shown in Figure 2.3.

Note that B_M is isomorphic to B_N , but there is no join preserving map between L_M and L_N . The labeled elements which are not in the respective Betti posets in both figures are incompatible.

In [5], we study a natural generalization of generic ideals, the class of rigid ideals. A rigid ideal can have non-simplicial multidegrees which correspond to unique multigraded basis elements in the minimal free resolution. To proceed, recall the class of rigid monomial ideals.

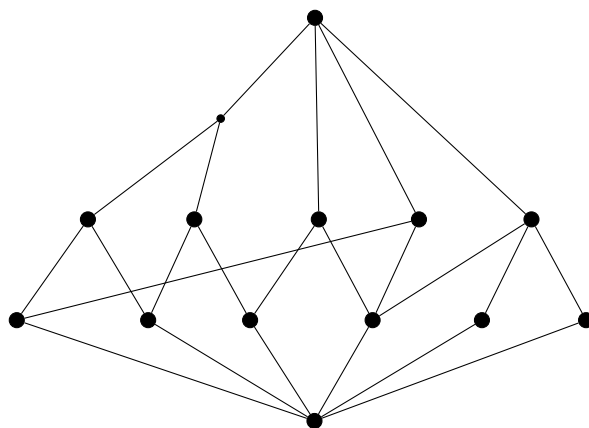
Definition 2.4. [5] Let M be a monomial ideal, with lcm-lattice L_M . Then M is a *rigid ideal* if the following two conditions hold:

- (R1) For every $p \in L_M$, we have $h_i(\Delta_p) = 1$ for at most one i .
- (R2) If there exist $p, q \in L_M$, where $h_i(\Delta_p) = h_i(\Delta_q) = 1$ for some i then p and q are incomparable in L_M .

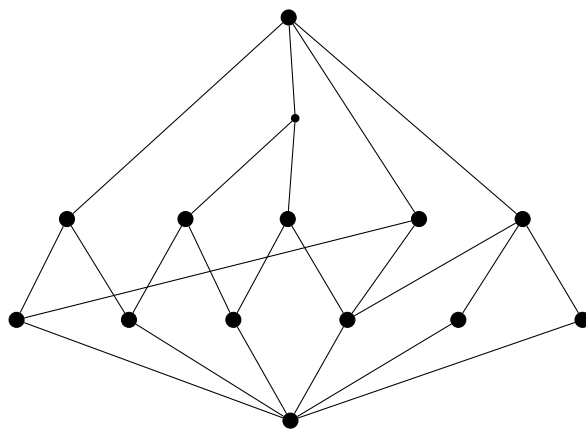
As one might expect, the rigidity of a monomial ideal is dependent on the characteristic of the field \mathbb{k} .

Remark 2.5. One implication of the definition of rigidity is that if \mathbf{b} and \mathbf{b}' are such that $\beta_{i,\mathbf{b}}$ and $\beta_{j,\mathbf{b}'}$ are nonzero and $\mathbf{b} > \mathbf{b}'$ in L then $i > j$. To see this assume $j > i$. Since \mathbf{b}' is the \mathbb{Z}^d -degree of a j th syzygy then there must be a $(j-1)$ st syzygy whose \mathbb{Z}^d -degree divides \mathbf{b}' (i.e. it will be less than \mathbf{b}' in L). Repeating this we obtain a chain of elements in L which ends in an element corresponding to an i th syzygy of \mathbb{Z}^d -degree $\mathbf{c} < \mathbf{b}'$ in L . This contradicts rigidity condition (R2), since by transitivity $\mathbf{c} < \mathbf{b}$ in L and both $\beta_{i,\mathbf{b}}$ and $\beta_{i,\mathbf{c}}$ are nonzero. Hence, $i > j$.

Figure 1: Lattices from Example 2.3



(a) L_M with B_M marked



(b) L_N with B_N marked

We are in a position to state the second commutative algebra result of this paper.

Theorem 2.6. *The Betti poset supports the minimal free resolution of a rigid monomial ideal.*

The proof of Theorem 2.6 amounts to showing that a sequence of vector spaces derived from the combinatorial structure of the Betti poset B_M is an exact complex. We postpone the details of this technical argument to Section 3.

In the remainder of this section, we discuss the significance of Theorem 2.6 with regards to a progression of ideas concerning the minimal resolution of monomial ideals. We use examples to make the case for the importance of the class of rigid ideals as fundamental to the construction of minimal resolutions.

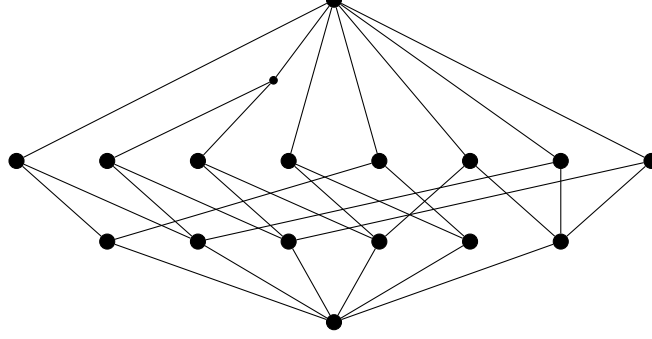
Remark 2.7. Note that one can always modify the minimal resolution of an arbitrary ideal by scaling or permuting the \mathbb{Z}^d -graded basis vectors and propagating this change in order to preserve exactness. Furthermore, for non-rigid ideals one may also change the \mathbb{Z}^d -graded basis in ways which change the (combinatorial) structure of the minimal free resolution. In the case of rigid ideals, such a change is impossible. In particular, we characterize rigid ideals in [5, Proposition 1.5] as having a minimal free resolution with unique \mathbb{Z}^d -graded basis. Theorem 2.6 therefore provides a unique combinatorial object which encodes both the unique \mathbb{Z}^d -graded basis and the mapping structure of the minimal free resolution of a rigid ideal.

Theorem 2.6's combinatorial prescription for the minimal resolution of a rigid monomial ideal allows us to take aim at Kaplansky's original question. For a non-rigid monomial ideal M we strive to find a *rigid deformation* of M whose resolution can be relabeled to give a resolution of M . We use the word deformation here as a reference to the notion of a *generic deformation* in [1] and [11]. Our notion of producing a rigid deformation will not involve perturbing the exponents of the ideal's generators, so we omit an explicit discussion of deformations of exponents as mentioned in [1] and [11]. Instead we will discuss the equivalent notion found in [8] and [9]. First we will need the following definition.

Definition 2.8. [14] Let $\mathcal{L}(n)$ be the set of all finite atomic lattices with n ordered atoms. Set $P \geq Q \in \mathcal{L}(n)$ if there exists a join preserving map $f : P \rightarrow Q$ which is a bijection on atoms.

The condition that there be a join preserving map which is a bijection on atoms, is the same condition found in Theorem 3.3 of [8] which states that a minimal resolution of an ideal with lcm-lattice P can be relabeled to be a resolution of an ideal with lcm-lattice Q . In [8] the authors note that the deformation of exponents from [1] produces lcm-lattices with this join preserving map. Moreover, [9, Theorem 5.1] states that for any two comparable finite atomic lattices in $\mathcal{L}(n)$, there exist monomial ideals so that the ideal whose

Figure 2: The lcm-lattice L_I of Example 2.11



lcm-lattice is P is a deformation of exponents of the ideal whose lcm-lattice is Q .

Definition 2.9. A monomial ideal J is a *rigid deformation* of the monomial ideal I if J is a rigid ideal, and the resolution of J can be relabeled to minimally resolve I .

Note here we do not simply require comparability in $\mathcal{L}(n)$ as would be suggested by the work of [8] and [9]. Corollary 2.10 and Example 2.11 should illuminate to the reader why we chose to make the definition this way.

Corollary 2.10. Let L be a finite atomic lattice and suppose that $P > L \in \mathcal{L}(n)$ is rigid with the same Betti numbers as L . Then every L' for which $B(L) \cong B(L')$ has its minimal resolution supported on $B(P)$. In other words, P is a rigid deformation of L' .

Example 2.11. Let $R = \mathbb{k}[a, s, t, u, v, w, x, y, z]$ and consider the squarefree monomial ideal $I = (uvxyz, atwxyz, stuwz, astuvw, asuvwxy, stvyz)$ whose lcm-lattice is pictured in Figure 2. The rigid deformation of I has an lcm-lattice which is not comparable to L_I in $\mathcal{L}(6)$, although the minimal free resolution of I is supported on a regular CW-complex with the intersection property. In particular, the highlighted element in L_I is not in B_I . Any attempt at a rigid deformation that produces a lattice comparable to L_I forces an increase in total Betti numbers. To produce a rigid deformation of I , we locate a lattice which is comparable to B_I , the lattice created from L_I by removing the highlighted element removed. We are fortunate that B_I is also a finite atomic lattice.

We have the following result guaranteeing the existence of a rigid deformation for certain monomial ideals.

Proposition 2.12. If I is a monomial ideal whose minimal free resolution is supported on a simplicial complex X , then there exists a rigid ideal J whose minimal free resolution is also supported on X such that $L_I < L_J \in \mathcal{L}(n)$. That is, J is a simplicial rigid deformation of I .

Proof. For ease of notation let us establish the following, $L = L_I$, and P is the augmented face poset of simplices of X , making P a finite atomic lattice. Note that each of these lattices have the same number of atoms, and that in terms of atomic supports, certain elements of L correspond to elements in P . In what follows we will denote elements of L which do not also correspond to elements in P as l (possibly indexed), elements of P as p (possibly indexed). Our goal is to construct a new lattice T with the properties that T is greater than L and P in $\mathcal{L}(n)$ such that there is an equality of total Betti numbers $\beta(T) = \beta(P)$. This lattice T will give rise to the monomial ideal J (and in fact L_J will be isomorphic to T).

Thinking of our finite atomic lattices as sets of sets (where the sets correspond to atomic supports of each of the elements in the lattice), let T be the meet closure of $L \cup P$. By construction, T is a finite atomic lattice. Moreover, we can think of T as consisting of elements of the following type $\{l \mid l \in L \text{ and does not correspond to an element of } P\}$ and $\{p \mid p \in P\}$. Note that one might worry that there is a subset of elements in T which takes the form $\{m \mid m = l \wedge p\}$. This is however, not the case. Since X is a simplicial complex, for each $p \in P$ every subset of the atomic support of p in P corresponds to a distinct element in P .

We need to show the following.

1. $\tilde{h}_i(\Delta_l^T) = 0$ for all i .
2. $\tilde{h}_i(\Delta_p^T) = \tilde{h}_i(\Delta_p^P)$ for all i .

Proving the second item is easy. Since X is a simplicial complex, none of the elements l can be less than any element p . This means all of the open intervals $(\hat{0}, p)$ in T are isomorphic to the same interval in P , guaranteeing that the homology groups are the same.

To prove the first item, we first assume l is not greater than any other elements of type l . In other words, all of the elements less than l are of the type p . In this case the open interval $(\hat{0}, l)$ in T is isomorphic to the union of half closed intervals

$$P_{\leq l} = \bigcup_{p_i \leq_T l} (\hat{0}, p_i]_P.$$

By the acyclicity condition in [2], we know that $X_{\leq l}$ is acyclic for every $l \in L$. Since $\Delta(P_{\leq l})$ is the barycentric subdivision of $X_{\leq l}$ we conclude that $\tilde{h}_i(\Delta_l^T) = 0$ for all i .

Now we need to remove the assumption that l is not greater than any elements of the type l . We do this by working up from the bottom of the lattice T . We want to find an l satisfying the earlier assumption, by the above argument $\tilde{h}_i(\Delta_l^T) = 0$ for all i . By Lemma 1.1 we can delete l to create a subposet T' so any $l' > l$ in T no longer has l below them in T' and the homology computations for intervals in T' are the same as in T . Now if l was the only element of type

l less than l' in T , then in T' the element l' satisfies the assumption that it is only greater than elements of type p . Hence, the previous argument applies so that $\tilde{h}_i(\Delta_l^T) = \tilde{h}_i(\Delta_{l'}^{T'}) = 0$ for all i . Note that in iterating this process we reduce T down to P (or $P - \{\hat{1}\}$ if X is not a simplex), guaranteeing that $\beta(T) = \beta(P)$. \square

We have reason to believe that a more general statement is true and propose the following conjecture. Recall that a CW-complex is said to have the intersection property if the intersection of any two cells is also a cell.

Conjecture 2.13. *If I is a monomial ideal with a minimal resolution supported on a regular CW-complex with the intersection property then I admits a rigid deformation.*

The notion of rigidity is not limited to resolutions supported on topological objects. In fact, Velasco's example of an ideal with non-CW resolution [17] is a rigid ideal. However, the assumption in Conjecture 2.13 that the regular CW-complex satisfy the intersection property is necessary. Consider the following example, whose homological structure was pointed out to us by Adam Boocher.

Example 2.14. Let I be the edge ideal of the hexagon,

$$I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_1x_6).$$

This ideal's resolution is supported on a three-dimensional regular CW-complex whose f -vector is $(1, 6, 9, 6, 2)$. This complex is absent the intersection property. Direct calculation shows that adding any single element to the lcm-lattice L_I increases Betti numbers. Hence, there is no finite atomic lattice with the same Betti numbers whose Betti poset is isomorphic to B_I . Therefore, I does not admit a rigid deformation.

In this example, the key step which does not allow us to proceed along the lines of the previous proof is that for the CW-complex supporting the resolution, the meet-closure of the poset of cells is isomorphic to L_I . Hence, no distinct lattice P exists, and we cannot construct the corresponding lattice T .

3 The minimal resolution of a rigid ideal

In order to prove Theorem 2.6, we first describe the construction found in [4], which produces a sequence of vector spaces and maps using the data of a finite poset P . For $\ell \geq 1$, the vector spaces in this sequence are

$$\mathcal{D}_\ell = \bigoplus_{p \in P \setminus \{\hat{0}\}} \tilde{H}_{\ell-2}(\Delta_p, \mathbb{k}).$$

In the border case, $\mathcal{D}_0 = \tilde{H}_{-1}(\{\emptyset\}, \mathbb{k})$, a one-dimensional \mathbb{k} -vector space. Note that the atoms (level one elements) of P index the nontrivial components of the vector space \mathcal{D}_1 .

These vector spaces are yoked into a sequence

$$\mathcal{D}(P) : \cdots \rightarrow \mathcal{D}_\ell \xrightarrow{\varphi_\ell} \mathcal{D}_{\ell-1} \rightarrow \cdots \rightarrow \mathcal{D}_1 \xrightarrow{\varphi_1} \mathcal{D}_0,$$

by maps φ_i whose structure comes from simplicial topology. To be precise, denote the order complex of a half closed interval $(\hat{0}, p]$ as $\Delta_{[p]}$. We have the following decomposition of the order complex of the open interval $(\hat{0}, q)$

$$\Delta_q = \bigcup_{p \triangleleft q} \Delta_{[p]}.$$

Focusing on the homological interaction between a fixed $\Delta_{[p]}$ and the rest of Δ_q , set

$$\Delta_{q,p} = \Delta_{[p]} \cap \left(\bigcup_{\substack{b \triangleleft q \\ b \neq p}} \Delta_{[b]} \right).$$

and note that $\Delta_{q,p} \subset \Delta_p$.

The maps φ_i are defined componentwise for every p using the connecting maps in the Mayer-Vietoris sequence for the triple

$$\left(\Delta_{[p]}, \bigcup_{\substack{b \triangleleft q \\ b \neq p}} \Delta_{[b]}, \Delta_q \right).$$

Specifically, $\varphi_i : \mathcal{D}_i \rightarrow \mathcal{D}_{i-1}$ is defined as $\varphi_i|_{\mathcal{D}_{i,q}} = \sum_{p \triangleleft q} \varphi_i^{q,p}$. The compo-

nent map $\varphi_i^{q,p} = \iota \circ \delta_{i-2}^{q,p} : \mathcal{D}_{i,q} \rightarrow \mathcal{D}_{i-1,p}$, is the composition of the map on homology induced by inclusion, $\iota : \tilde{H}_{i-3}(\Delta_{q,p}, \mathbb{k}) \rightarrow \tilde{H}_{i-3}(\Delta_p, \mathbb{k})$ with $\delta_{i-2}^{q,p} : \tilde{H}_{i-2}(\Delta_q, \mathbb{k}) \rightarrow \tilde{H}_{i-3}(\Delta_{q,p}, \mathbb{k})$, the connecting map from the Mayer-Vietoris sequence.

We proceed with the proof by first showing that $\mathcal{D}(B_M)$ is a complex when M is rigid.

For a poset element $q \in P$, write $\ell(q) = \max\{j : p_1 \triangleleft p_2 \triangleleft \cdots \triangleleft p_j = q\}$ for the level of q , the maximum possible length of a saturated chain ending in q . As a consequence, $\ell(\hat{0}) = 0$ and the atoms of P are of level 1. The rank of q is $\text{rank}(q) = 1 + \ell(q)$ and may be thought of as the maximum number of poset elements (including q) which appear within a saturated chain which ends at q . For $q \in P$, write $\mathcal{D}(P)_{\leq q}$ for the subsequence of $\mathcal{D}(P)$ constructed by using the half-closed interval $(\hat{0}, q]$ with maps given as restrictions of the maps from $\mathcal{D}(P)$.

Definition 3.1. Given $q \in P$ let C be the set of all chains $c \in (\hat{0}, q]$ such that c has $\ell(q)$ elements. The *maximal ranked subposet of* $(\hat{0}, q]$ is the set

$$\text{MR}(P, q) = \{p \in (\hat{0}, q] : p \in c \in C\},$$

with comparison inherited from P .

Remark 3.2. By definition, $\text{MR}(B_M, q)$ is ranked for all $q \in B_M$. Using Proposition 7.1 from the Appendix of [4], the sequence $\mathcal{D}(\text{MR}(B_M, q))$ is a complex. Moreover, in the case when B_M itself is ranked, then by the same proposition, $\mathcal{D}(B_M)$ is a complex.

Remark 3.3. Note that Remark 2.5 implies that for a rigid monomial ideal, if one records the number i for which $\beta_{i,p}$ is nonzero at each element p in $\text{MR}(B_M, q)$ then the i strictly increase along chains. Specifically, $i = 0$ at the minimal element, and i increases by one, traveling cover by cover along chains in $\text{MR}(B_M, q)$.

Our goal is to use the fact in the previous remark to show that $\mathcal{D}(B_M)$ is a complex even when B_M is not ranked. We aim to do this by showing that the last two maps in any multigraded strand $\mathcal{D}(B_M)_{\leq q}$ behave exactly like the maps coming from a ranked poset. The following lemma shows that the free modules we obtain by passing to $\text{MR}(B_M, q)$ will agree in the last two spots of the complex.

Lemma 3.4. *Let M be rigid and $\tilde{h}_t(\Delta_q^{B_M}, k) = 1$ for a specific t . For $p \in \text{MR}(B_M, q)$ such that $\tilde{h}_i(\Delta_p^{B_M}, k) = 1$ for $i = t, t-1$, or $t-2$ we obtain the equivalence*

$$\tilde{h}_i(\Delta_p^{B_M}, k) = \tilde{h}_i(\Delta_p^{\text{MR}(B_M, q)}, k)$$

for all $i > t-2$.

Proof. If $p \in \text{MR}(B_M, q)$ then $p \in B_M$. By rigidity of M , let us say that $\tilde{h}_i(\Delta_p^{B_M}, k) = 1$ and $\tilde{h}_j(\Delta_p^{B_M}, k) = 0$ for $j \neq i$. This means that the maximal chains in $(\hat{0}, p)_{B_M}$ are of length $i+1$ and correspond to the faces forming a cycle w_p which generates $\tilde{H}_i(\Delta_p^{B_M}, k)$. As these are maximal chains and since $p \in \text{MR}(B_M, q)$ we can see that these are the same as the maximal chains in $(\hat{0}, p)_{\text{MR}(B_M, q)}$. Thus w_p also generates homology for $\tilde{H}_i(\Delta_p^{\text{MR}(B_M, q)}, k)$.

We will use the Mayer-Vietoris sequence, and for simplicity designate $X = \Delta_p^{B_M}$, $A = \Delta_p^{\text{MR}(B_M, q)}$, and define the poset $B' = \{r \in B_M \mid r \notin \text{MR}(B_M, q), r < p\}$ and let $B = \Delta(B')$. By construction we see that $X = A \cup B$. By rigidity, we know that $\tilde{H}_i(X, k)$ is nonzero for only one i (which by the assumptions of the lemma is either $t, t-1$, or $t-2$) and is zero for all other i . Moreover, we can also see that by rigidity that $\tilde{H}_i(B, k)$ is zero for all i , since if it were not (say it was nonzero for some j) then $\tilde{H}_i(X, k)$ would also be nonzero in homological j which would be a contradiction (see the definition of rigidity and Corollary 1.3 in [5]). Thus by Mayer-Vietoris we get the following isomorphisms from the long exact sequence:

1. $\tilde{H}_i(A, k) \cong \tilde{H}_i(X, k)$ because $\tilde{H}_i(A \cap B, k)$ is necessarily zero since the dimension of $A \cap B$ is strictly less than the dimension of B which is less than i .
2. $\tilde{H}_j(A \cap B, k) \cong \tilde{H}_j(A, k)$ for $j \neq i$ since $\tilde{H}_j(X, k)$ is zero for all $j \neq i$.

In the second case we want to show that these groups are both zero for $j = t$, or $t - 1$. For $j > i$, both are clearly zero since the dimension of A is i (regardless of how i relates to t). For $j < i$, we need only show that the isomorphism holds for $j = t$, or $t - 1$. Here the maximal chains of B' are of at most length i since chains in $(\hat{0}, p)_M^B$ are of at most length $i + 1$. Moreover as the maximal chains of B' are not maximal chains of $\text{MR}(B_M, q)$ and since $p \notin B'$, this further limits the maximal length of chains in B' to be of at most length $i - 1$. Thus in the simplicial complex $A \cup B$ the maximum dimension of any face is $i - 2$. Now consider our limits on what values i can take. If $i = t$, then the dimension of $A \cup B$ is $t - 2$ and the homology vanishes in homological degrees t and $t - 1$. If $i = t - 1$, the dimension of $A \cup B$ is $t - 3$ and so the homology vanishes in the appropriate places. And the same is true if $i = t - 2$. \square

The next lemma demonstrates certain components of the maps in $\mathcal{D}(B_M)_{\leq q}$ are zero.

Lemma 3.5. *If M is rigid and $p \in B_M$ is covered by q , and $p \notin \text{MR}(B_M, q)$ then $\varphi_i^{q,p}$ in $\mathcal{D}(B_M)$ is the zero map for all i .*

Proof. Because $p, q \in B_M$ we know that there exist $k, l \in \mathbb{Z}$ such that $\tilde{h}_k(\Delta_p, \mathbb{k}) = 1$ and $\tilde{h}_l(\Delta_q, \mathbb{k}) = 1$.

First we need to show that if $p < q$ in B_M and $p \notin \text{MR}(B_M, q)$ then $l - k > 1$. Clearly $l - k \neq 0$ for if so, we would contradict rigidity condition (R2). Moreover by Remark 2.5, (R2) also implies that $l > k$.

Given that $l > k$, it remains to show that $l > k + 1$. We see that the subposet $(\hat{0}, q]$ in $\text{MR}(B_M, q)$ must contain a maximal chain of length $l + 1$ (in fact all chains in $\text{MR}(B_M, q)$ have the same length which is $l + 1$). So the elements $p \in B_M$ which are not elements of $\text{MR}(B_M, q)$ must lie in a chain of largest possible length which is less than $l + 1$. Hence, $(\hat{0}, p]$ must contain a maximal chain of length $k + 1$. Since $p < q$, the maximum length of a chain in $(\hat{0}, q]$ which contains p must be $k + 2$. Thus, $k + 2 < l + 1$, or equivalently $k + 1 < l$, as claimed.

Now we can finally show that $\varphi_i^{q,p} = 0$. We will do this by showing that

$$\delta_{i-2}^{q,p} : \tilde{H}_{i-2}(\Delta_q, \mathbb{k}) \rightarrow \tilde{H}_{i-3}(\Delta_{q,p}, \mathbb{k})$$

is zero for all i . Since M is rigid, if Δ_q only has nonzero homology in homological degree l then we need only focus our attention on the case when $i - 2 = l$ since otherwise the map is already zero. Specifically, we must show that $\tilde{H}_{l-1}(\Delta_{q,p}, \mathbb{k}) = 0$.

First, note that $\Delta_{q,p}$ can also be expressed as the order complex of the poset

$$(B_M)_{q,p} := (\hat{0}, p] \cap \left(\bigcup_{\substack{p' \leq q \\ p' \neq p}} (\hat{0}, p'] \right).$$

Necessarily, $(B_M)_{q,p} \subset (\hat{0}, p)$ because $p \notin (\hat{0}, p')$ for any such $p' \triangleleft q$. This means that the length of the longest chain in $(B_M)_{q,p}$ is less than or equal to k and furthermore, that the maximum possible dimension of a face in $\Delta_{q,p}$ is $k - 1$. So if $i > k - 1$ it must be that $\tilde{h}_i(\Delta_{q,p}, \mathbb{k}) \neq 0$. As previously argued, we know that $l > k + 1$ so that $k - 1 < k < l - 1$. Thus $\tilde{H}_{l-1}(\Delta_{q,p}, \mathbb{k}) = 0$. \square

Now we will show that the last two maps in $\mathcal{D}(B_M)_{\leq q}$ and $\mathcal{D}(\text{MR}(B_M, q))$ are the same.

Lemma 3.6. *The length of $\mathcal{D}(B_M)_{\leq q}$ is equal to the length of $\mathcal{D}(\text{MR}(B_M, q))$. Moreover, writing this common length as l , the first homological degree where the two sequences can possibly differ is $l - 2$.*

Proof. Denote $\text{MR}(B_M, q) - \{q\}$ as $\text{MR}(B_M, \hat{q})$ and write $\Delta_{\text{MR}, q}$ for the order complex $\Delta(\text{MR}(B_M, \hat{q}))$. To see that the length of both sequences is the positive integer l , it suffices to show that

$$\tilde{h}_{l-2}(\Delta_{\text{MR}, q}, \mathbb{k}) = \tilde{h}_{l-2}(\Delta_q, \mathbb{k}).$$

This follows immediately from the fact that the maximal chains of the ranked poset $\text{MR}(B_M, \hat{q})$ are exactly the maximal chains of $(\hat{0}, q)$, and that these chains correspond to the $(l - 2)$ -faces of the respective order complexes.

To see that the last two maps are the same in the sequences $\mathcal{D}(B_M)_{\leq q}$ and $\mathcal{D}(\text{MR}(B_M, q))$ first observe that if $B_M = \text{MR}(B_M, q)$ then the two sequences of maps are identical. Thus it remains to consider the case when $(\hat{0}, q] \subset B_M$ is not ranked. Lemma 3.5 guarantees that φ_l is identical in both sequences. Moreover, we know that the basis elements of \mathcal{D}_{l-1} (in the sequence $\mathcal{D}(B_M)_{\leq q}$) must correspond to elements $p \in \text{MR}(B_M, q)$ where $p \triangleleft q$. Indeed if this were not the case consider an element p' covered by q in B_M which is not in $\text{MR}(B_M, q)$. By Remark 3.3, the Betti number $\beta_{i,p'}$ must be nonzero for some $i < l - 1$ since p' is in a chain of length less than $l + 1$. Thus, the free modules in position $l - 1$ are the same in both sequences.

Finally we will show that the maps φ_{l-1} are also the same. Note by 3.4, it is possible for the free modules in position $l - 2$ to differ between the two sequences, but the domain for φ_{l-1} in both sequences are the same. Moreover, if there are basis elements in position $l - 2$ for the sequence $\mathcal{D}(B_M)_{\leq q}$ which are not in $\mathcal{D}(\text{MR}(B_M, q))$ then they must correspond to elements $p' \in B_M$ such that $p' \notin \text{MR}(B_M, q)$. Since p' is in a non-maximal chain this means that p' is not comparable to any of the elements p corresponding to basis elements in position $l - 1$ since they are all in $\text{MR}(B_M, q)$. Thus $\varphi_{l-1}^{p,p'} = 0$ for all $p \in \text{MR}(B_M, q)$ corresponding to basis elements in position $l - 1$. Moreover this implies that the only elements in $(B_M)_{\leq q}$ contributing nonzero components of φ_{l-1} correspond to elements in $\text{MR}(B_M, q)$. Thus the maps φ_{l-1} are the same between the two sequences. \square

With these lemmas in hand we now prove that $\mathcal{D}(B_M)$ is a complex.

Theorem 3.7. *$\mathcal{D}(B_M)$ is a complex.*

Proof. First note that if B_M is ranked then $\mathcal{D}(B_M)$ is a complex by Proposition 7.1 in the Appendix of [4].

Assume that B_M is not ranked. We need to show that for any element $a \in \mathcal{D}_i$, we have $\varphi_{i-1}(\varphi_i(a)) = 0$. It is sufficient to check this for basis elements. Let $q \in B_M$ correspond to a basis element in \mathcal{D}_i . To verify that $\varphi_{i-1}(\varphi_i(q)) = 0$, it is sufficient to check the composition of the maps in $\mathcal{D}(B_M)_{\leq q}$. By Lemma 3.6 we see that this criteria can be checked using the corresponding maps in $\mathcal{D}(\text{MR}(B_M, q))$. However, since $\text{MR}(B_M, q)$ is ranked then, Proposition 7.1 of [4] implies that $\mathcal{D}(\text{MR}(B_M, q))$ is a complex. Thus $\varphi_{i-1}(\varphi_i(q)) = 0$, as needed. \square

With $\mathcal{D}(B_M)$ shown to be a complex, it remains to verify exactness. In the arguments that follow, square brackets written around a simplicial chain always denote a homology class.

Theorem 3.8. *If M is a rigid monomial ideal, then $\mathcal{D}(B_M)$ is exact.*

Proof. Write p for the projective dimension of the rigid module R/M and fix $1 \leq i \leq p$. The construction of $\mathcal{D}(B_M)$ guarantees that an element $v \in \ker(\varphi_i)$ has the structure of a finite sum $v = \sum c_q \cdot [w_q]$, where $q \in B_M$, the coefficient $c_q \in \mathbb{k}$ and $[w_q]$ is a generator for the vector space $\tilde{H}_{i-2}(\Delta_q, \mathbb{k})$.

Since v is an abstract sum of homology classes, each coming from a unique vector space summand, $v = 0$ if and only if $c_q = 0$ for every q . In such a scenario, $v \in \text{im}(\varphi_{i+1})$. Therefore, in order to prove exactness, we must show that every nonzero kernel element v is an element of $\text{im}(\varphi_{i+1})$.

Since the sum v is built from the homology classes of $\tilde{H}_{i-2}(\Delta_q^{B_M}, \mathbb{k})$ for various $q \in B_M$, we appeal to the structure of B_M as a subposet of L_M and to the relationship between the simplicial homology of the relevant open intervals in these posets. In particular, Theorem 1.4 guarantees that the class $[w_q]$ also generates $\tilde{H}_{i-2}(\Delta_q^{L_M}, \mathbb{k})$.

As preparation for the rest of the proof, recall the notion of a cone of a simplicial complex taken over a disjoint apex a . This complex consists of simplices which are simplicial joins of the apex a with simplices σ of the original simplicial complex. We write $\{a, \sigma\}$ for the associated simplicial chain in a cone complex.

Define $u = \sum c_q \cdot \{q, w_q\}$ to be the finite sum of simplicial chains in the algebraic chain complex $\mathcal{C}(\Delta(B_M))$. The chains in this sum are created by coning the chains representing the classes $[w_q]$ over the apex q . The sum u is an oriented simplicial chain in $\Delta(B_M)$ and the inclusion of posets $B_M \subseteq L_M$ guarantees that our sum is also an oriented chain of $\Delta(L_M)$. Since L_M is a lattice, then there exists $y \in L_M$ such that $y = \vee q$ for those q appearing in the sum.

Case 1: Suppose $\tilde{H}_{i-1}(\Delta_y^{L_M}, \mathbb{k}) \neq 0$, so that $y \in B_M$. Such a y covers every $q \in B_M$ which appears in the sum u . Indeed, any $x \in B_M$ with the property that $q < x < y$ must have $\tilde{H}_j(\Delta_x^{L_M}, \mathbb{k}) \neq 0$ for $i-2 < j < i-1$, which is impossible.

If the class $[u] = \left[\sum c_q \cdot \{q, w_q\} \right]$ generates $\tilde{H}_{i-1}(\Delta_y^{L_M}, \mathbb{k})$, then it certainly generates $\tilde{H}_{i-1}(\Delta_y^{B_M}, \mathbb{k})$ since $[u]$ is defined using simplices of $\Delta(B_M)$. This is

not necessarily the case if we take an arbitrary generator of the homology of $\Delta_y^{L_M}$.

Applying the map φ_{i+1} directly to $[u]$, we have

$$\begin{aligned}
\varphi_{i+1}([u]) &= \varphi_{i+1} \left(\left[\sum c_q \cdot \{q, w_q\} \right] \right) \\
&= \sum c_q \cdot \varphi_{i+1}|_{\mathcal{D}_{i,q}}(\{q, w_q\}) \\
&= \sum c_q \cdot \left(\sum_{q \leq y} \varphi_{i+1}^{y,q}(\{q, w_q\}) \right) \\
&= \sum c_q \cdot \left(\sum_{q \leq y} \iota \circ \delta_{i-1}^{y,q}(\{q, w_q\}) \right) \\
&= \sum c_q \cdot \left(\sum_{q \leq y} [d_{i-1}(\{q, w_q\})] \right) \\
&= \sum c_q \cdot \left(\sum_{q \leq y} [w_q] + [q, d_{i-2}(w_q)] \right) \\
&= \sum c_q \cdot \left(\sum_{q \leq y} [w_q] + [q, 0] \right) \\
&= \sum c_q \cdot [w_q] = v,
\end{aligned} \tag{3.1}$$

where $[q, d_{i-2}(w_q)] = [q, 0]$ since we assumed that $[w_q]$ is a cycle in Δ_q . Thus, $v \in \text{im}(\varphi_{i+1})$.

If the class $[u] = \left[\sum c_q \cdot \{q, w_q\} \right]$ does not generate $\tilde{H}_{i-1}(\Delta_y^{L_M}, \mathbb{k})$, then it certainly cannot generate $\tilde{H}_{i-1}(\Delta_y^{B_M}, \mathbb{k})$. There are now two possibilities.

If $[u]$ generates $\tilde{H}_j(\Delta_y^{L_M}, \mathbb{k})$ in some dimension $j \neq i-1$, then we have found a y which contributes in dimensions j and $i-1$, which contradicts rigidity condition (R1).

Suppose that $[u] = \left[\sum c_q \cdot \{q, w_q\} \right]$ is zero in $\tilde{H}_{i-1}(\Delta_y^{L_M}, \mathbb{k})$. Thus, writing $u = \sum c_q \cdot \{q, w_q\}$ we have $d(u) = d\left(\sum c_q \cdot \{q, w_q\}\right)$ is the boundary of a simplicial chain in $\Delta_y^{B_M} \subset \Delta_y^{L_M}$. Calculating directly, we see that

$$d(u) = d\left(\sum c_q \cdot \{q, w_q\}\right) = \sum c_q \cdot \{w_q\} - \sum c_q \{q, d(w_q)\}$$

must bound. We have assumed $[d(u)] = 0$, so that passing to homology classes,

$$0 = [d(u)] = \left[d\left(\sum c_q \cdot \{q, w_q\}\right) \right] = \left[\sum c_q \cdot \{w_q\} \right] - \left[\sum c_q \{q, d(w_q)\} \right].$$

However w_q is a homology cycle for every q , so $d(w_q) = 0$ and therefore,

$$0 = [d(u)] = \left[\sum c_q \cdot \{w_q\} \right] = \sum c_q \cdot [w_q].$$

Thus, we have shown that our kernel element $v = \sum c_q \cdot [w_q]$ must be equal to zero in this subcase and therefore $v \in \text{im}(\varphi_{i+1})$

Case 2: Suppose $\tilde{H}_{i-1}(\Delta_y^{L_M}, \mathbb{k}) = 0$, so that $y \notin B_M$.

If the class $[u] = \left[\sum c_q \cdot \{q, w_q\} \right]$ generates $\tilde{H}_{i-1}(\Delta_x^{L_M}, \mathbb{k})$ for some x , then $x < y \in L_M$ and $x \in B_M$. If such an x did not exist but $[u]$ generated homology then $y \in B_M$, contradicting our assumption that $y \notin B_M$.

In fact, such an $x \in B_M$ which has $[u]$ as a generator for $\tilde{H}_{i-1}(\Delta_x^{L_M}, \mathbb{k})$ must be unique. To see this, suppose that x and x' are two elements of B_M whose open interval homology is generated by $[u]$. Then $q < x$ and $q < x'$ for every q in the sum. There are now two possibilities. First, if $x < x'$ (or $x > x'$), then we would have found two comparable elements in B_M having nonzero Betti numbers in the same homological degree. This is a contradiction to rigidity condition (R2).

On the other hand, if x and x' are incomparable, then within B_M , there exist at least two elements a and b such that $a < x$ and $a \not< x'$, while $b < x'$ and $b \not< x$. (If these elements did not exist then either $x = x'$ or they are comparable.) Since M is rigid and $[u]$ has been assumed to be a nontrivial homology class, then any elements a and b which are present in the order complexes $\Delta_x^{B_M}$ and $\Delta_{x'}^{B_M}$ need not be part of any chain which generates the respective homologies. Indeed, were a or b part of such a chain, then one possibility is that such a chain generates homology separately from $[u]$. However, this contradicts rigidity condition (R1), since we assume that $[u]$ already generates a one-dimensional space without containing chains ending in a or b . Alternately, if this new chain was homologous to $[u]$, and generated the homology of one of the order complexes $\Delta_x^{B_M}$ or $\Delta_{x'}^{B_M}$, but not the other, then we have a contradiction to our assumption that $[u]$ generates homology for both order complexes.

Furthermore, $x \wedge x' \notin B_M$ when x and x' are incomparable elements of B_M which are both comparable to all the q 's. Indeed, were this meet to exist in B_M , then a class generating the homology of the order complexes $\Delta_x^{B_M}$ and $\Delta_{x'}^{B_M}$ would not be carried by chains containing the q 's. In the lcm-lattice L_M , however, the meet $x \wedge x'$ must exist. Using the comparability relation in L_M , then certainly $x \wedge x' > q$ for every q which indexes a summand in the original definition of the class $[u]$. However, since $x \wedge x' \notin B_M$ then by definition, $\tilde{H}_j(\Delta_{x \wedge x'}^{L_M}, \mathbb{k}) = 0$ for every j . Hence, the homology of $\Delta_x^{L_M}$ and $\Delta_{x'}^{L_M}$ cannot be carried solely by the subcomplex $\Delta_{[x \wedge x']}$. However, any oriented chain which only contains poset chains ending in q 's is carried by this subcomplex. Thus, were x and x' distinct, then $[u]$ could not generate homology in the asserted dimension.

Having established that when $[u]$ is a nontrivial homology cycle, it generates the homology of $\Delta_x^{B_M}$ for exactly one $x \in B_M$, we now apply the map φ_{i+1} . This calculation is similar to the one detailed in (3.1), where the covering element in this case is x . The associated equation implies that $v \in \text{im}(\varphi_{i+1})$.

Our final possibility is that the class $[u]$ does not generate $\tilde{H}_{i-1}(\Delta_x^{B_M}, \mathbb{k})$ for any $x \in B_M$. Together with the assumption that the $[w_q]$ generate homology in

dimension $i - 2$, rigidity guarantees that the class $[u]$ cannot generate homology in any dimension $j \neq i - 1$. Hence, either $[u] = 0$, or $[u]$ generates $i - 1$ dimensional homology of $\Delta(B_M)$ since the elements q would be maximal in B_M . The former case implies that $v = 0$, which was argued in Case 1. We claim that the latter case is impossible due to the fact that for any poset P with a maximum element, the Betti poset $B(P \setminus \{\hat{0}\})$ is acyclic.

Towards verification of this claim, write $\hat{1}$ for the maximum element of P . If $\hat{1} \in B(P)$, then the order complex $\Delta(B(P \setminus \{\hat{0}\}))$ is a cone with apex $\hat{1}$, and is acyclic.

On the other hand, if $\hat{1} \notin B(P)$, then we proceed by induction on the number of non-contributing elements of P . If $\hat{1}$ is the unique non-contributing element of P , then $B(P \setminus \{\hat{0}\})$ must be acyclic. Indeed, if we assume otherwise, the equality $B(P \setminus \{\hat{0}\}) = (\hat{0}, \hat{1}) \subset P$ would imply that the order complex $\Delta(\hat{0}, \hat{1}) = \Delta(B(P \setminus \{\hat{0}\}))$ had nontrivial homology, contradicting our assumption that $\hat{1}$ is a non-contributor.

Suppose that $k > 1$ and that for any poset P' with k non-contributors, the Betti poset $B(P' \setminus \{\hat{0}\})$ is acyclic. Let P be a poset with $k + 1$ non-contributors. For any non-contributor $x \neq \hat{1}$, Corollary 1.3 guarantees an equality of Betti posets $B(P \setminus \{\hat{0}\}) = B(P \setminus \{\hat{0}, x\})$. By the inductive hypothesis, $P \setminus \{x\}$ has k non-contributors, so that $B(P \setminus \{\hat{0}, x\})$ is acyclic. Hence, $B(P \setminus \{\hat{0}\})$ is acyclic as claimed. Since an lcm-lattice L_M has a maximum element, then $B(L_M \setminus \{\hat{0}\})$ is acyclic.

This completes the proof of exactness. \square

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